

# ON HOMALOIDAL POLYNOMIALS

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Let  $\mathbb{P}^n$  be the projective space over a field of characteristic zero. If  $F$  is a homogeneous polynomial we say that  $F$  is homaloidal if the polar map  $\partial F$  defined by the partial derivatives of  $F$  is a birational selfmap of  $\mathbb{P}^n$ . Although the problem of determining homaloidal polynomials has a classical flavour the theme only recently was raised in an algebro-geometric context by Dolgachev ([Do]), following suggestions stemming from the theory of prehomogeneous varieties: the relative invariants of prehomogeneous spaces are in fact homaloidal polynomials ([KiSa],[EKP]). Dolgachev classifies homaloidal polynomials in  $\mathbb{P}^2$  (see also [Di]) and characterizes homaloidal polynomials in  $\mathbb{P}^3$  which are products of linear forms as products of four independent linear forms. Dolgachev also raises the question if it is true that a non square free product of linear forms is homaloidal if and only the product of its factors with multiplicity one is. This question has been given a positive answer in a specific case (see [KS]) and in full generality ([DiPa]) in a topological context.

We will give an algebraic proof of the following:

**Theorem A.** *Suppose  $F = \prod_{i=0}^r L_i$  is a square free homaloidal polynomial which is the product of linear forms. Then  $r = n$  and the linear forms  $L_0, \dots, L_n$  are independent, so that  $\partial F$  is a composition of a projectivity and of a standard Cremona transformation.*

**Theorem B.** *Suppose  $F = \prod_{i=0}^r L_i^{n_i}$  is a homogeneous polynomial with  $L_0, \dots, L_r$  linear forms. Then  $F$  is homaloidal if and only if the polynomial  $F_{red} = \prod_{i=0}^r L_i$  is homaloidal. In particular, if  $F$  is homaloidal, then  $r = n$ , the linear forms  $L_0, \dots, L_n$  are independent and the map  $\partial F$  is a composition of projectivities and of a standard Cremona transformation.*

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**1.Preliminaries** We start with the following classical:

**Definition 1** If  $F \in H^0(\mathbb{P}^n, \mathcal{O}_{P^n}(d))$  is a homogeneous polynomial, the *polar map defined by  $F$*  is the rational map defined by

$$\partial F : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n*}$$

$$\partial F(p) = [\frac{\partial F}{\partial X_0}(p), \dots, \frac{\partial F}{\partial X_n}(p)],$$

while the *polar system defined by  $F$*  is the linear system

$$|\partial F| := |< \frac{\partial F}{\partial X_0}, \dots, \frac{\partial F}{\partial X_n} >| \subset |\mathcal{O}_{P^n}(d-1)|.$$

It follows from the definition that, if  $Z_F \subset \mathbb{P}^n$  is the hypersurface defined by  $F$ , the base locus of the polar map defined by  $F$  is the singular locus of  $Z_F$ . Moreover, since  $Z_F$  is an hypersurface,  $Z_F$  is not reduced if and only if  $F$  is not square free. In case  $F$  is square free the polar map  $\partial F$  is then free of base divisors and if  $Z_F$  is smooth the polar map  $\partial F$  is a morphism whose image is the dual variety  $Z_F^* \subset \mathbb{P}^{n*}$  of  $Z_F$ .

If  $F$  is not square free, let us write

$$F = \prod_{i=0}^r G_i^{n_i},$$

with  $d = \sum_{i=0}^r n_i \deg G_i$ . Then the base components of the polar system  $|\partial F|$  are given by the hypersurface defined by the polynomial

$$F' = \prod_{i=0}^r G_i^{n_i-1}.$$

We will indicate by  $F_{red}$  the polynomial  $\frac{F}{F'}$ .

The polar system defined by  $F$  is naturally split in a fixed and in a moving part:  
**Definition 2** The *moving part* of the polar system defined by a homogeneous polynomial  $F$  is the linear system  $|M(\partial F)|$  obtained by removing all base components from the polar system  $|\partial F|$ .

In particular, we have that

$$|M(\partial F)| \subset |\mathcal{O}_{\mathbb{P}^n}(d-1-\deg F')|.$$

Notice that the above definition makes perfectly sense in case  $F$  is square free, in which case  $F'$  is a constant.

**Definition 3** A *homaloidal polynomial* of degree  $d$  is a homogeneous polynomial  $F \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$  such that the moving part  $|M(\partial F)|$  of the polar system defined by  $F$  induces a *birational map*.

Well known examples of homaloidal polynomials in  $\mathbb{P}^n$  are those defining smooth quadrics. A remarkable result in [EKP] is the classification of homaloidal polynomials of degree  $d = 3$ : the irreducible ones define the secant varieties of the four Severi varieties and a classification seems at hand at least in degree  $d = 4$ . Probably the most known and important example of homaloidal polynomial is the polynomial  $F = X_0 \cdots X_n$ , which has degree  $d = n + 1$  and whose associated polar map is a *standard Cremona transformation*. Another class of examples of homaloidal polynomials is given by the polynomials

$$F(m_0, \dots, m_n) := X_0^{m_0} \cdots X_n^{m_n},$$

with  $m_i \geq 1$ , for all  $i = 0, \dots, n$ , of arbitrary large degree  $d = \sum_{i=0}^n m_i$ . The base locus of  $\partial F$  has a divisorial component defined by the polynomial  $F' = \prod_{i=0}^n X_i^{m_i-1}$ . Once we remove it, we simply compute that

$$|M(\partial F)| = |< m_0 \prod_{i=1}^n X_i, \dots, m_j \prod_{i \neq j} X_i, \dots, m_n \prod_{i=0}^{n-1} X_i >|,$$

so that  $\partial F$ , induces the same map as a composition of a projectivity (a homothety) and  $\partial F_{red} = \partial \prod_{i=0}^n X_i$ , so that it is homaloidal.

To say that  $F$  is homaloidal is equivalent to say that  $\partial F$  is dominant and that there exists a resolution of singularities

$$\begin{array}{ccc} & X & \\ f \swarrow & & \searrow g \\ \mathbb{P}^n - - \partial F - - & \rightarrow & \mathbb{P}^{n*}, \end{array}$$

such that if  $Y$  is a general member of  $|M(\partial F)|$  and if  $\overline{Y}$  denotes its strict transform on  $X$ , we have

$$(\overline{Y})^n = 1,$$

because in fact  $\overline{Y} \simeq g^* \mathcal{O}_{\mathbb{P}^{n*}}(1)$ , where  $\simeq$  denotes linear equivalence as Cartier divisors and  $g$  is a birational morphism.

An important property of homaloidal polynomials is the following:

**Proposition 4.** *If  $F$  is a homaloidal polynomial,  $Z_F$  is not a cone. In particular if  $F$  is the product of linear forms  $F = \prod_{i=0}^r L_i^{m_i}$ , then  $< L_0, \dots, L_r > = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ , so that in particular  $r \geq n$ .*

*Proof.* If  $Z_F$  is a cone the image of the polar map defined by  $F$  is contained in the linear space dual to the vertex of the cone of  $Z_F$ . If  $F$  is a product of linear forms  $Z_F$  is a cone if and only if  $\langle L_0, \dots, L_r \rangle \neq H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ .  $\square$

In fact (see [R]) even more is true:  $Z_F$  is a cone if and only if the image of the polar map associated to  $F$  lies in a hyperplane.

**2. Products of linear forms** In this section we will prove Theorems A and B. We will always assume that  $F$  is a homaloidal polynomial which splits in the product of linear forms. Suppose first that  $F$  is square free so that  $\partial F$  is a linear system free of base components. We will fix a minimal resolution of singularities:

$$\begin{array}{ccc} & X & \\ f \swarrow & & \searrow g \\ \mathbb{P}^n - \partial F & \dashrightarrow & \mathbb{P}^{n*}. \end{array}$$

By definition, if  $Y$  is a general member of the polar system  $|\partial F|$ , and if  $\bar{Y}$  is its strict transform on  $X$ , we have  $\bar{Y} \simeq g^* \mathcal{O}_{\mathbb{P}^{n*}}(1)$ .

Let us write

$$F = \prod_{i=0}^r L_i.$$

Up to a projectivity we can assume that  $L_0 = X_0$ . We will denote by  $H_0$  the hyperplane defined by  $X_0 = 0$ . Let us define the polynomial

$$G := \frac{F}{X_0} = \prod_{i=1}^r L_i.$$

With this choice, a basis of the polar system  $|\partial F|$  defined by  $F$  is given by:

$$|\partial F| = \left| \left\langle G + X_0 \frac{\partial G}{\partial X_0}, X_0 \frac{\partial G}{\partial X_1}, \dots, X_0 \frac{\partial G}{\partial X_n} \right\rangle \right|.$$

We first observe that, if  $D \subset X$  is the strict transform of the hyperplane  $H_0$ , looking at the equations for  $\partial F$ , the map  $g$  contracts  $D$ , because  $\partial F$  contracts  $H_0$  to its dual point  $U_0 = [1 : 0 : \dots : 0]$ .

Let us define  $G_0 \in H^0(H_0, \mathcal{O}_{H_0}(d-1))$  as the restriction of  $G$  to  $H_0$ ; notice that  $G_0$  doesn't need to be reduced.

**Lemma 5.** *The irreducible divisor  $D \subset X$  is the unique divisor contracting to the point  $U_0 \in \mathbb{P}^{n*}$  and the map  $g : X \rightarrow \mathbb{P}^{n*}$  factors through the blow up  $h' : Z \rightarrow \mathbb{P}^{n*}$  of  $\mathbb{P}^{n*}$  at  $U_0$ .*

*Proof.* Suppose that  $W \neq D$  is a divisor in  $X$  which is  $g$ -exceptional and such that  $g(W) = g(D) = U_0$ . By minimality of the resolution of the rational map  $\partial F$ ,  $W$  is not  $f$ -exceptional, so that it corresponds to a hypersurface  $f(W) \subset \mathbb{P}^n$  which is distinct from  $H_0$ . Let  $J$  be an equation of  $f(W)$ . Looking at the equations of  $\partial F$ , it follows that  $J$  must divide  $\frac{\partial G}{\partial X_i}$  for all  $i \geq 1$ . If we restrict the system  $|\langle \frac{\partial G}{\partial X_1}, \dots, \frac{\partial G}{\partial X_n} \rangle|$  to the hyperplane  $H_0$ , by analyticity of polynomials, we obtain nothing but the polar system defined by  $G_0$  on  $H_0$ . This implies that  $J$  restricted to  $H_0$  is a base component of  $|\partial G_0|$  and this means that

$$f(W) \cap H_0 \subset \cup \{L_i \mid \text{there exists } L_j \text{ for which } L_j \in \langle H_0, L_i \rangle\}.$$

But the polar map  $\partial F$  contracts each one of the hyperplanes defined by the linear forms  $L_i$  to their dual points in  $\mathbb{P}^{n*}$ , so that it cannot be  $g(W) = g(H_0)$ .

The irreducible divisor  $D$  corresponds then to the extraction of a valuation centered at  $U_0 \in \mathbb{P}^{n*}$  and we need to show that this valuation corresponds to the whole maximal ideal  $m_{U_0}$ . We have already noticed en passant that the system  $|\langle \frac{\partial G}{\partial X_1}, \dots, \frac{\partial G}{\partial X_n} \rangle|$  corresponds on  $X$  to the system  $|g^* \mathcal{O}_{\mathbb{P}^{n*}}(1) - D|$  and that it is of codimension one in  $|g^* \mathcal{O}_{\mathbb{P}^{n*}}(1)|$ . Suppose that  $g_* \mathcal{O}_X(-D) = m'$  with  $\sqrt{m'} = m_{U_0}$ ; since

$$g_* g^* \mathcal{O}_{\mathbb{P}^{n*}}(1) \otimes \mathcal{O}_X(-D) = m' \otimes \mathcal{O}_{\mathbb{P}^{n*}}(1),$$

we have that  $H^0(\mathbb{P}^{n*}, m' \otimes \mathcal{O}_{\mathbb{P}^{n*}}(1)) = n$  and hence that

$$m_{U_0} = m'.$$

Hence  $D$  is the strict transform of the exceptional divisor under the blow up of  $\mathbb{P}^{n*}$  at  $U_0$ ,  $h' : Z \rightarrow \mathbb{P}^{n*}$  and the result follows  $\square$

Consider now the following diagram of maps:

$$\begin{array}{ccc} & X & \\ f \swarrow & & \searrow g \\ H_0 \subset \mathbb{P}^n - - \partial F - - & \rightarrow & \mathbb{P}^{n*} - \pi - \rightarrow P, \end{array}$$

where  $\pi$  is the projection from the point  $U_0$  to the hyperplane  $P$ .

We can use Lemma 5 in order to factorize the morphism  $g$  through the blow up  $Z$  of  $\mathbb{P}^{n*}$  at  $U_0$ . We have the following diagram:

$$\begin{array}{ccc} & X & \\ h \downarrow & & \\ & Z & \\ h' \swarrow & & \searrow t \\ \mathbb{P}^{n*} - \pi - - & \rightarrow & P, \end{array}$$

with  $g = h' h$ . Let us denote by  $Y'$  a general member of the polar system  $|\partial G|$  and recall that we denote by  $Y$  a general element in  $|\partial F|$  and by  $D$  the strict transform of  $H_0$  in  $X$ .

**Lemma 6.** *With the above notations:*

- (1) *the composition  $th : X \rightarrow P$  is a morphism and  $(th)^* \mathcal{O}_P(1) \simeq \overline{Y'} \simeq \overline{Y} - D$ ,*
- (2) *the restriction of the linear system  $|(th)^* \mathcal{O}_P(1)|$  to  $D$  induces a morphism  $m : D \rightarrow P$  which is a resolution of singularities of the polar map associated to  $G_0$  on  $H_0$ , i.e.  $|(\overline{Y} - D)|_D| = |M(\partial G_0)|$ ,*
- (3) *the polynomial  $G_0$  is homaloidal in  $H_0$ ,*
- (4)  *$G_0$  is square free if and only there don't exist  $L_i$  and  $L_j$ , with  $i \neq j$ , such that  $X_0 \in \langle L_i, L_j \rangle$*

*Proof.*

It follows from the definitions that the morphism  $th$  is set theoretically the same as the rational map  $\pi g$ , so that in fact  $th$  is a resolution of singularities of  $\pi g$ : they define the same morphism up to removing base components from the linear system defining  $\pi g$ , which is in fact  $D$ . Now, the linear system defining  $th$  is  $|\overline{Y} - D| = |(th)^* \mathcal{O}_P(1)| = |g^* \mathcal{O}_{\mathbb{P}^{n*}}(1) - D|$ ; by the above argument, this is also the moving part of  $|\partial G|$ , so that in fact  $Y' \simeq \overline{Y} - D$ . It follows from the explicit equations that if we restrict to  $H_0$  the map  $\pi g$ , the resulting restriction is the rational map  $\partial G_0$ , whose resolution of singularities is the map  $th$  restricted to  $D$ . We just notice here that the linear system inducing  $m := th|_D$  is  $|(\overline{Y} - D)|_D| = |M(\partial G_0)|$ . The map  $\partial G_0$  is surjective because a composition of surjections. In order to show that  $G_0$  is homaloidal on  $H_0$  it suffices to show that

$$D \cdot ((th)^* \mathcal{O}_P(1) - D)^{n-1} = 1.$$

This follows from the fact that  $D$  is the strict transform of the exceptional divisor under the blow up  $h' : Z \rightarrow \mathbb{P}^{n*}$ . Finally, since the base components of  $|\partial G_0|$  on  $H_0$  are non reduced components of  $Z_{G_0}$ , and since  $G$  is square free, the last part of the thesis is proved.  $\square$

**Remark** As a result of our Theorems A and B it will follow that  $Z_G$  is a cone. If one is able to prove this directly, the proof of Theorem A follows easily and directly from Lemma 6 (see Theorem 8). It is very easy to show that  $Z_G$  is a cone if and only if  $th$  is in fact the full polar map  $\partial G$  if and only if in  $\mathbb{P}^{n*}$  the point corresponding to  $H_0$  is not on a line connecting  $Z_{L_i}^*$  and  $Z_{L_j}^*$  for  $i \neq j \neq 0$  if and only if the polynomial  $G_0$  is square free. This is the hard part of the problem, connected with cohomological properties of the corresponding arrangement of hyperplanes in  $\mathbb{P}^n$  and with syzygies of the base locus of  $\partial F$ . It is in fact remarkable that Dolgachev proves it directly in case  $n = 3$ .

Let us now consider homaloidal polynomials which are products of linear forms with at least a square factor; notice that such polynomials arise naturally from

square free homaloidal polynomials which are products of linear forms: in Lemma 6 we have proven that the restriction of a homaloidal square free product of linear forms induces on each component of  $Z_F$  a homaloidal product of linear forms. Quite surprisingly *there is a priori no relation among  $\partial F$  and  $\partial F_{red}$* .

Let us choose in fact  $r + 1$  distinct linear forms  $L_0, \dots, L_r$  in  $\mathbb{P}^n$  together with an identification  $X_0 = L_0$ , and consider the following polynomials, where  $H_0$  is the hyperplane of equation  $X_0 = 0$  and  $L_{i,0} = L_i \cap H_0$ :

$$\begin{aligned} F &= X_0^{m_0} \prod_{i=1}^r L_i^{m_i}, & F' &= X_0^{m_0-1} \prod_{i=1}^r L_i^{m_i-1}, & F_{red} &= \frac{F}{F'}, \\ G &= \prod_{i=1}^r L_i^{m_i}, & G' &= \prod_{i=1}^r L_i^{m_i-1}, & G_{red} &= \frac{G}{G'}, \\ G_0 &= G \cap H_0, & G'_0 &= G' \cap H_0. \end{aligned}$$

We compute the moving parts of the polar systems defined by  $F$  and  $F_{red}$ . We have:

$$\begin{aligned} |M(\partial F)| &= < m_0 G_{red} + X_0 \sum_{i=1}^r m_i \frac{\partial L_i}{\partial X_0} \prod_{j \neq i,0} L_j, \dots, X_0 \sum_{i=0}^r m_i \frac{\partial L_i}{\partial X_n} \prod_{j \neq i,0} L_j > |, \\ |M(\partial F_{red})| &= |\partial F_{red}| = < G_{red} + X_0 \sum_{i=1}^r \frac{\partial L_i}{\partial X_0} \prod_{j \neq i,0} L_j, \dots, X_0 \sum_{i=0}^r \frac{\partial L_i}{\partial X_n} \prod_{j \neq i,0} L_j > |. \end{aligned}$$

We also compute an explicit basis of a system  $|G''_0|$  which sits in a chain  $|M(\partial G_0)| \subset |G''_0| \subset |\partial G_0|$  on  $H_0$ :

$$|G''_0| = < \sum_{i=1}^r m_i \frac{\partial L_i}{\partial X_1} \prod_{j \neq i} L_{j,0}, \dots, \sum_{i=1}^r m_i \frac{\partial L_i}{\partial X_n} \prod_{j \neq i} L_{j,0} > |,$$

Consider now the following diagram of maps, where  $f$  and  $g$  induce a minimal resolution of the morphism induced by  $|M(\partial F)|$ :

$$\begin{array}{ccc} & X & \\ f \swarrow & & \searrow g \\ H_0 \subset \mathbb{P}^n & \xrightarrow{-M(\partial F)} & \mathbb{P}^{n*} \xrightarrow{-\pi} P. \end{array}$$

We define  $D$  to be the strict transform of  $H_0$  in  $X$ , we denote by  $Y$  a general member of  $|M(\partial F)|$ , by  $Y'$  a general member of the system  $|M(\partial G)|$ , by  $\overline{Y}$  and  $\overline{Y}'$  their strict transforms on  $X$ . Quite surprisingly, all the arguments used in order to prove Lemma 5 and Lemma 6 apply verbatim and in particular it holds the following:

**Lemma 7.** *With the above notations:*

- (1)  $D$  is the strict transform on  $X$  of the exceptional divisor in the blow up  $h' : Z \rightarrow \mathbb{P}^{n*}$  of  $\mathbb{P}^{n*}$  at  $U_0$ ,
- (2) the restriction of the linear system  $|(th)^*\mathcal{O}_P(1)|$  to  $D$  induces a morphism  $m : D \rightarrow P$  which is a resolution of singularities of the polar map defined by  $G_0$  on  $H_0$ , i.e.  $|(\bar{Y} - D)|_D| = |M(\partial G_0)|$ ,
- (3) the polynomial  $G_0$  is homaloidal in  $H_0$ ,

*Proof.*  $\square$

We are now able to prove Theorems A and B at once.

**Theorem 8.** *Let  $L_0, \dots, L_r$  be distinct linear forms and let  $F = \prod_{i=0}^r L_i^{m_i}$ , with  $m_i \geq 1$  for all  $i = 0, \dots, r$ . Then  $F$  is homaloidal if and only if  $F_{red} = \prod_{i=0}^r L_i$  is homaloidal and  $F_{red}$  is homaloidal if and only if  $r = n$  and the  $L_i$ 's are independent linear forms.*

*Proof.* The proof is by induction on  $n$ .

The starting point of the induction is the case  $n = 1$  which is easy: if  $F = \prod_{i=0}^r L_i^{m_i}$  is homaloidal the base point free system  $|M(\partial F)|$  must be of degree one, from which it follows easily that  $r = 1$  and that  $L_0$  and  $L_1$  are in linear general position (they are distinct by hypothesis). The converse is, up to a projectivity, been proved after Definition 3. The same argument works a fortiori if  $F$  is square free.

Let us then move to  $\mathbb{P}^n$ , with  $n > 1$  and consider first the case of a square free homaloidal polynomial  $F = \prod_{i=0}^r L_i$ . We plug  $X_0 = L_0$ , and we apply Lemma 6 in order to get a homaloidal polynomial  $G_0 = \prod_{i=1}^r L_{i,0}$  on  $H_0$ . If  $G_0$  is reduced we have that by induction  $r = n$  and the  $L_{i,0}$ 's are independent in  $H_0$ , from which it follows that  $X_0, L_1, \dots, L_n$  are independent in  $\mathbb{P}^n$ . Consider then the case in which  $G_0$  is not reduced.  $G_0$  is still homaloidal so that by induction we can reorder the  $L_i$ 's in such a way that  $L_{1,0}, \dots, L_{n,0}$  are independent and there exist  $m_1, \dots, m_n$  for which  $\sum_{i=1}^n m_i = r$  and

$$G_0 = \prod_{i=1}^n L_{i,0}^{m_i},$$

in such a way that

- (1)  $L_{n+1}, \dots, L_{n+m_1-1}$  are in  $\langle X_0, L_1 \rangle$ ,
- (2) ..
- (3)  $L_{n+m_1+\dots+m_{n-1}-n+2}, \dots, L_{n+\sum_{i=1}^n m_i-n}$  are in  $\langle X_0, L_n \rangle$ .

In other words, looking at the dual points in  $\mathbb{P}^{n*}$ , we must have that all  $Z_{L_i}^*$ 's, with  $i \geq n+1$  must lie in the cone with vertex  $U_0$  projecting  $Z_{L_1}^*, \dots, Z_{L_n}^*$ . But we can apply the same reasoning we have applied to  $L_0 = X_0$  to any other  $L_i = X_i$  for all  $i = 1, \dots, n$  and the intersection of all these cones is empty, so that there will



exists some  $i \in \{0, \dots, n\}$  for which the corresponding homaloidal polynomial  $G_i$  is reduced. We then proceed as if  $G_0$  were reduced.

Suppose now that  $F = \prod_{i=0}^r L_i^{m_i}$  is nonreduced and homaloidal. We must prove that  $F_{red}$  is homaloidal, the converse being a consequence of the first part of this Proof and of the example after Definition 3. Plugging  $X_0 = L_0$  and applying Lemma 7 we get that  $|M(\partial F)|$  induces on  $H_0$  the homaloidal system defined by  $G_0$ . By induction and by the same argument as above, we get the thesis, i.e. that  $r = n$  and the linear forms  $L_0, \dots, L_n$  are independent, so that  $F_{red}$  is homaloidal.

□

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